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On derivations and commuting like elements in prime rings

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Abstract

The purpose of this paper is to investigate some central differential identities involving a fixed element of R, more precisely, we will prove that, if a is a fixed element of R satisfying some special differential identities then a is central. Moreover, the classifications of the involved derivations are also provided.

Key words: Prime rings, Center of rings, Derivations, Commutativity.

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1 Introduction

Rings considered in this paper are associative and not necessarily unitary. We shall denote by $Z(R)$ the center of a ring R. An ideal P of R is a prime ideal if $xRy \subseteq P$ yields $x \in P$ or $y \in P$. In particular, if the zero ideal of R is prime, then R is said to be a *prime ring*. For any $x, y \in R$, we will write $[x, y] = xy - yx$ and $x \circ y = xy + yx$ for the Lie product and Jordan product, respectively. An additive mapping $d : R \longrightarrow R$ is a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$.
Over the last years, several authors have investigated the relationship between the commutativity, the structure of the ring R and certain special type of maps on R. We first recall that a mapping $f : R \to R$ is called *centralizing* on R, if $[f(x), x] \in Z(R)$ for all $x \in R$; in the special case where $[f(x), x] = 0$ for all $x \in R$, the mapping f is said to be *commuting* on R. In [12], Posner proved that if a prime ring R admits a nonzero centralizing derivation d on R, then R is commutative. Since then many authors have extended the Posner's result in several directions. A considerable number of researchers have investigated and proved that some subsets of a ring R, defined by certain sort of commutativity condition, coincide with its center $Z(R)$. In [6], Herstein showed that, if R is a ring with no nonzero nil ideal, then the *hypercenter*+

$$S(R) := \{a \in R \mid [a, x^n] = 0 \text{ for all } x \in R \text{ and an integer } n \geq 1\}$$

coincides with the center $Z(R)$ of R. Motivated by Herstein's hypercenter, Chacron [4] introduced a more general concept that he called the *cohyper-center* $T(R)$ of R defined by

$$T(R) := \{a \in R \mid [a, x - x^2 p(x)] = 0 \text{ for all } x \in R \text{ and } p(x) \in \mathbb{Z}[x] \text{ depends on } (a, x)\}.$$

He proved that the cohypercenter of a semiprime ring R is exactly the center of R. In [9], Herstein introduced the set

$$H(R; d) := \{a \in R \mid ad(x) = d(x)a \text{ for all } x \in R\},$$

where d is a nonzero derivation on R, he was been able to prove that if R is a $2-$torsion free prime ring then $H(R; d) = Z(R)$.
Recently, Idrissi et al. [10], introduced and studied the following new center-like subsets:

$$Z^{+}(R, d) := \{y \in R \mid [d(x), d(y)] + [x, y] \in Z(R) \text{ for all } x \in R\}.$$

$$Z^{-}(R, d) := \{y \in R \mid [d(x), d(y)] - [x, y] \in Z(R) \text{ for all } x \in R\}.$$

$$Z^{*-}(R, d) := \{y \in R \mid [d(x), d(y)] - [y, d(x)] - [d(y), x] \in Z(R) \text{ for all } x \in R\},$$

where d is a derivation of R, they actually proved that if R is a 2-torsion free prime ring then $Z^{-}(R, d) = Z^{+}(R, d) = Z(R)$. Moreover, if $d \neq 0$ then $Z^{*-}(R, d) = Z(R)$.
In this paper, we continue the investigation about theses subsets by studying the behavior of a fixed element $a \in R$ satisfying some differential identities in prime rings.

2 The Main results

Before starting the proofs, we need to recall the following well-known facts.

Fact 1. *[11, Theorem 1] Let R be a 2-torsion free noncommutative prime ring, a a nonzero element of R, d a nonzero derivations of R such that $[a, d(R)] \subset Z(R)$, then $a \in Z(R)$.*

Fact 2. *[13, Lemma 1] Let R be a semiprime ring and $a, b, c \in R$. If $axb + bxc = 0$ for all $x \in R$ then $(a + c)xb = 0$ for all $x \in R$.*

Fact 3. *Let R be a prime ring. If $ab \in Z(R)$ and $a \in Z(R)$ then $a = 0$ or $b \in Z(R)$.*

It is proved in [1], that if R is a 2-torsion free prime ring, d a nonzero derivation and a an element of R satisfying $d([x, a]) \in Z(R)$ for all $x \in R$, then a is a central element. In the next Theorem, we investigate a more general identity with two derivations.

Theorem 1. *Let R be a 2-torsion free noncommutative prime ring, a a nonzero element of R, d_1 and d_2 are nonzero derivations of R such that $d_1(xa) - d_2(ax) \in Z(R)$ for all $x \in R$, then one of the following assertions holds:*

1. *$a \in Z(R)$ and $d_1 = d_2$;*
2. *There exists $\lambda \in \mathcal{C}$ such that $d_1(x) = \lambda[x, a]$ and $d_2(x) = \lambda[a, x]$ for all $x \in R$ and $a^2 \in Z(R)$.*

Proof. Suppose that

$$d_1(xa) - d_2(ax) = 0 \text{ for all } x \in R. \tag{1}$$

Replacing x by xa, we get

$$xad_1(a) - axd_2(a) = 0 \text{ for all } x \in R. \tag{2}$$

Writing ux instead of x in the above equation, with $u \in R$, we get

$$[u, a]xd_2(a) = 0 \text{ for all } x, u \in R. \tag{3}$$

Hence, $a \in Z(R)$ or $d_2(a) = 0$. If $a \in Z(R)$, Eq. (1) becomes

$$(d_1 - d_2)(ax) = 0 \text{ for all } x \in R. \tag{4}$$

Which leads to $d_1+d_2 = 0$. Now, suppose that $a \notin Z(R)$ then $d_2(a) = 0$ and the hypothesis becomes

$$d_1(xa) - ad_2(x) = 0 \text{ for all } x \in R. \tag{5}$$

Now, replace x by ax in Eq. (5), to obtain $d_1(a) = 0$, then the equation reduces to

$$d_1(x)a - ad_2(x) = 0 \text{ for all } x \in R. \tag{6}$$

Writing ux instead of x in the last equation and using it, we get

$$d_1(u)[x, a] + [u, a]d_2(x) = 0 \text{ for all } x, u \in R. \tag{7}$$

Now, instead of u we put xu in the last equation and we use it to get

$$d_1(x)u[x, a] + [x, a]ud_2(x) = 0 \text{ for all } x, u \in R. \tag{8}$$

Hence, by Fact (2) we have $d_1 + d_2 = 0$. Invoking Eq. (6), we find that

$$d_1(x) \circ a = 0 \text{ for all } x \in R. \tag{9}$$

Replacing x by xu in the last relation and using ([3], Lemma 2.2), there exists $\lambda \in \mathcal{C}$ such that $d_1(x) = \lambda[x, a]$ for all $x \in R$ and so $d_2(x) = \lambda[a, x]$ for all $x \in R$. Remark that Eq. (9) leads to $a^2 \in Z(R)$.
Now, if $Z(R) = (0)$, then from the first part of this proof, we conclude the desired result. So, assume that $Z(R) \neq (0)$

$$d_1(xa) - d_2(ax) \in Z(R) \text{ for all } x \in R. \tag{10}$$

Let $z(\neq 0) \in Z(R)$. Putting xz instead of x in Eq. (10), we get

$$xad_1(z) - axd_2(z) \in Z(R) \text{ for all } x \in R. \tag{11}$$

Replacing x by z, we get

$$az(d_1 - d_2)(z) \in Z(R) \text{ for all } x \in R. \tag{12}$$

Which yields to $a \in Z(R)$ or $(d_1 - d_2)(z) = 0$ for all $z(\neq 0) \in Z(R)$.
If $a \in Z(R)$, then Eq. (10) becomes

$$a(d_1 - d_2)(x) + x(d_1 - d_2)(a) \in Z(R) \text{ for all } x \in R. \tag{13}$$

Commuting with x, we get

$$a[(d_1 - d_2)(x), x] = 0 \text{ for all } x \in R. \tag{14}$$

Which leads to $d_1 = d_2$.
Now, suppose that $a \notin Z(R)$, then $(d_1 - d_2)(z) = 0$ for all $z(\neq 0) \in Z(R)$ thus Eq. (11) becomes

$$[x, a]d_1(z) \in Z(R) \text{ for all } x \in R. \tag{15}$$

Which means $d_1(z) = 0$, and so $d_2(z) = 0$. Then, by replacing x by z in Eq. (10), we get $(d_1 - d_2)(a) \in Z(R)$. Now, we replace x by xa in Eq. (10), we have

$$d_1(xa)a + (xa)d_1(a) - d_2(ax)a - (ax)d_2(a) \in Z(R) \text{ for all } x \in R. \tag{16}$$

Then, we replace x by ax in the same equation, and we have

$$d_1(a)(xa) + ad_1(xa) - d_2(ax)a - (ax)d_2(a) \in Z(R) \text{ for all } x \in R. \tag{17}$$

By combining the last two equations, and using Eq. (10) commuted with a, we find

$$[xa, d_1(a)] - [ax, d_2(a)] \in Z(R) \text{ for all } x \in R. \tag{18}$$

Since $(d_1 - d_2)(a) \in Z(R)$, the last equation becomes

$$[[x, a], d_2(a)] \in Z(R) \text{ for all } x \in R. \tag{19}$$

Which yields to $d_2(a) \in Z(R)$ and so $d_1(a) \in Z(R)$. Then, hypothesis becomes

$$d_1(x)a - ad_2(x) + x(d_1 - d_2)(a) \in Z(R) \text{ for all } x \in R. \tag{20}$$

We commute the last equation with $r \in R$ to get

$$[d_1(x)a, r] - [ad_2(x), r] + [x, r](d_1 - d_2)(a) = 0 \text{ for all } x, r \in R. \tag{21}$$

By putting xr instead of x and using Eq. (21), we get

$$[d_1(x)ra, r] - [d_1(x)ar, r] + x[d_1(r)a, r] + [x, r]d_1(r)a - [axd_2(r), r] = 0 \tag{22}$$

for all $x, r \in R$. From Eq. (20), we have $x[d_1(r)a, r] - x[ad_2(r), r] = 0$, and Eq. (22) becomes

$$[d_1(x)[r, a], r] + [[x, a]d_2(r), r] + [x, r](d_1(r)a - ad_2(r)) = 0 \tag{23}$$

for all $x, r \in R$. We replace r by a, and we get

$$[[x, a], a]d_2(a) + [x, a](d_1 - d_2)(a)a = 0 \text{ for all } x \in R. \tag{24}$$

Replacing r by rx in Eq. (24), and using the same equation we find

$$2[r, a][x, a]d_2(a) = 0 \text{ for all } x, r \in R. \tag{25}$$

Which yields to $d_2(a) = 0$, and with Eq. (24) we have $d_1(a) = 0$. Then, Eq. (20) becomes

$$d_1(x)a - ad_2(x) \in Z(R) \text{ for all } x \in R. \tag{26}$$

Replacing x by xa, we get

$$(d_1(x)a - ad_2(x))a \in Z(R) \text{ for all } x \in R. \tag{27}$$

It means

$$d_1(x)a - ad_2(x) = 0 \text{ for all } x \in R. \tag{28}$$

Replacing x by xr in Eq. (28) and using it, we get

$$d_1(x)[r, a] + [r, a]d_2(r) = 0 \text{ for all } x, r \in R. \tag{29}$$

Then, we replace x by rx in the last equation to find

$$d_1(r)x[r, a] + [r, a]xd_2(r) = 0 \text{ for all } x, r \in R. \tag{30}$$

It yields by Fact (2) to $d_1 + d_2 = 0$, then Eq. (28) becomes

$$d_1(x) \circ a = 0 \text{ for all } x \in R. \tag{31}$$

Replacing x by xr in Eq. (31) and using it, we get

$$[x, a]d_1(r) = d_1(x)[r, a] \text{ for all } x, r \in R. \tag{32}$$

In view of ([3]Lemma 2.2), there exists $\lambda \in \mathcal{C}$ such that $d_1(x) = \lambda[x, a]$ and $d_2(x) = \lambda[a, x]$ for all $x \in R$. Then from Eq. (31), $a^2 \in Z(R)$.

∎

Theorem 2. Let R be a 2-torsion free prime ring, a a nonzero element of R, d_1 and d_2 are nonzero derivations of R such that $d_1(xa) - d_2(ax) - [x, a] \in Z(R)$ for all $x \in R$, then one of the following assertions holds:

1. $a \in Z(R)$ and $d_1 = d_2$;
2. There exists $\lambda \in \mathcal{C}$ such that $d_1(x) = \lambda[x, a]$ and $d_2(x) = \lambda[a, x]$ for all $x \in R$ and $a^2 \in Z(R)$.

Proof. Similar to Proof of Theorem 1. ∎

Lemma 1. *Let R be a 2-torsion free noncommutative prime ring and d a derivation on R. There is no noncentral element a of R such that $d([x, a]) + [d(x), a] \in Z(R)$ for all $x \in R$.*

Proof. Let a be a noncentral element of R, such that

$$d([x, a]) + [d(x), a] = 0 \text{ for all } x \in R. \tag{33}$$

If $Z(R) = (0)$, then replacing x by ux, we get

$$d(u)[x, a] + [u, a]d(x) = 0 \text{ for all } x, u \in R. \tag{34}$$

In view of ([7], Lemma 1.3.2), there exists $\lambda \in C$, the extended centroid of R, such that

$$d(x) = \lambda[x, a] \text{ for all } x \in R. \tag{35}$$

Using the last expression in Eq. (33), we get

$$[d(x), a] = 0 \text{ for all } x \in R. \tag{36}$$

Which leads to $a \in Z(R)$ in view of Fact 1, a contradiction.
Now, we assume that $Z(R) \neq (0)$, which means that

$$d([x, a]) + [d(x), a] \in Z(R) \text{ for all } x \in R. \tag{37}$$

Let z be a nonzero element of $Z(R)$. By replacing x by xz in Eq. (37), we get

$$[x,a]d(z) \in Z(R) \text{ for all } x \in R. \tag{38}$$

Which yields to $[x,a] \in Z(R)$ for all $x \in R$ or $d(z) = 0$. Hence $d(z) = 0$ for all $z \in Z(R)$. Replacing x by xa in Eq. (37), we get

$$2[x,a]d(a) + (d[x,a] + [d(x),a])a + x[d(a),a] \in Z(R) \text{ for all } x \in R. \tag{39}$$

And now replacing x by ax in Eq. (37), we get

$$2d(a)[x,a] + a(d[x,a] + [d(x),a]) + [d(a),a]x \in Z(R) \text{ for all } x \in R. \tag{40}$$

Using the two last equations, the fact that $[d(a),a] \in Z(R)$ and Eq. (37), we find that

$$2[d(a),[x,a]] \in Z(R) \text{ for all } x \in R. \tag{41}$$

In view of Fact (3), we have $a \in Z(R)$, which is a contradiction, or $d(a) \in Z(R)$. Hence, $d(a) \in Z(R)$ which means $d^2(a) = 0$. Now, by applying d on Eq. (39) we get

$$(3d([x,a]) + [d(x),a])d(a) = 0 \text{ for all } x \in R. \tag{42}$$

Since $(d([x,a]) + [d(x),a])d(a) \in Z(R)$ and R is 2-torsion free, we have

$$d[x,a]d(a) \in Z(R) \text{ for all } x \in R. \tag{43}$$

It yields that $d(a) = 0$ or $d[x,a] \in Z(R)$ for all $x \in R$, in both cases we have $[d(x),a] \in Z(R)$ for all $x \in R$, hence $a \in Z(R)$, a contradiction. ∎

Motivated by the results in [10], Ait Zemzami et al studied in [1] the behavior of a fixed element $a \in R$ satisfying $d([x,a]) - [x,a] \in Z(R)$ for all $x \in R$.

Theorem 3. Let R be a 2-torsion free prime ring, a a nonzero element of R, d_1 and d_2 are nonzero derivations of R such that $d_1([x,a]) + [d_2(x),a] \in Z(R)$ for all $x \in R$. Then $a \in Z(R)$.

Proof. Let a be a nonzero element of R. Suppose that

$$d_1([x,a]) + [d_2(x),a] \in Z(R) \text{ for all } x \in R. \tag{44}$$

In first time, assume that

$$d_1([x,a]) + [d_2(x),a] = 0 \text{ for all } x \in R. \tag{45}$$

Replacing x by xu, with $u \in R$, and using last equation we get

$$[x,a](d_1+d_2)(u) + (d_1+d_2)(x)[u,a] = 0 \text{ for all } x,u \in R. \tag{46}$$

Now, replacing u by ur and using Eq. 46, we find

$$[x,a]u(d_1+d_2)(r) + (d_1+d_2)(x)u[r,a] = 0 \text{ for all } x,u,r \in R. \tag{47}$$

Hence, in view of ([7], Lemma 1.3.2), there exists $\lambda \in \mathcal{C}$ such that

$$(d_1+d_2)(x) = \lambda[x,a] \text{ for all } x \in R. \tag{48}$$

Using the last equation, the hypothesis becomes

$$d_1([x,a]) - [d_1(x),a] + \lambda[[x,a],a] = 0 \text{ for all } x \in R. \quad (49)$$

Which implies that

$$[x, d_1(a)] + \lambda[[x,a],a] = 0 \text{ for all } x \in R. \quad (50)$$

Replacing x by xu, we find that

$$2\lambda[x,a][u,a] = 0 \text{ for all } x,u \in R. \quad (51)$$

Invoking the primeness of R we get $a \in Z(R)$.
Now, assume that Eq. (44) is satisfied. If $Z(R) = (0)$, then from the first part of this proof, we get the desired result.
So, suppose that $Z(R) \neq (0)$ and let $z \in Z(R) \setminus \{0\}$. Replacing x by xz in Eq. (44), we get

$$[x,a](d_1 + d_2)(z) \in Z(R) \text{ for all } x \in R. \quad (52)$$

So, $[x,a] \in Z(R)$ for all $x \in R$ or $(d_1 + d_2)(z) = 0$ for all $z \in Z(R)$. If $[x,a] \in Z(R)$ for all $x \in R$, then $a \in Z(R)$. Now, suppose $(d_1 + d_2)(z) = 0$ for all $z \in Z(R)$. Putting $h = d_1 + d_2$, the Eq. (44) becomes

$$[h(x),a] + [x, d_1(a)] \in Z(R) \text{ for all } x \in R. \quad (53)$$

Writing xa instead of x in Eq. (53), we obtain

$$[h(x),a]a + x[h(a),a] + [x,a]h(a) + x[a,d_1(a)] + [x,d_1(a)]a \in Z(R) \text{ for all } x \in R. \quad (54)$$

Then, writing ax instead of x in Eq. (53), we get

$$a[h(x),a] + [h(a),a]x + h(a)[x,a] + [a,d_1(a)]x + a[x,d_1(a)] \in Z(R) \text{ for all } x \in R. \quad (55)$$

Now, subtracting Eq. (55) from Eq. (54) and using Eq. (53), we find that

$$[[x,a],h(a)] \in Z(R) \text{ for all } x \in R. \quad (56)$$

Then, by Fact(1) $a \in Z(R)$ or $h(a) \in Z(R)$. In the last case, $h^2(a) = 0$ and from Eq. (53) $[a, d_1(a)] \in Z(R)$. By applying h to Eq. (54), and using the fact that $h(a) \in Z(R)$ we get

$$([h(x),a] + [x,d_1(a)])h(a) + h((x,a])h(a) + h(x)[a,d_1(a)] = 0 \text{ for all } x \in R. \quad (57)$$

Since $([h(x),a] + [x,d_1(a)])h(a) \in Z(R)$ for all $x \in R$, we have

$$[h(x),a]h(a) + h(x)[a,d_1(a)] \in Z(R) \text{ for all } x \in R. \quad (58)$$

Commuting the last expression with $h(x)[a,d_1(a)]$, we get

$$[[h(x),a],h(x)]h(a)[a,d_1(a)] = 0 \text{ for all } x \in R. \quad (59)$$

This leads to $[[h(x),a],h(x)] = 0$ for all $x \in R$, or $h(a) = 0$ or $[a,d_1(a)] = 0$, and in the three cases, we'll have $a \in Z(R)$. ∎

Now it is natural to ask, what is going to happen if we consider the Jordan product instead of the Lie product in the identity of the previous Theorem?

Theorem 4. *Let R be a 2-torsion free noncommutative prime ring, a a nonzero element of R, d_1 and d_2 are nonzero derivations of R such that $d_1(x \circ a) + d_2(x) \circ a \in Z(R)$ for all $x \in R$, then $a^2 \in Z(R)$.*

Proof. Let a be a nonzero element of R. If $Z(R) = (0)$, so the hypothesis becomes

$$d_1(x \circ a) + d_2(x) \circ a = 0 \quad \text{for all} \ \ x \in R. \tag{60}$$

Replacing x by ux, and using the hypothesis, we get

$$d_1(u(x \circ a)) - d_1([u, a]x) + d_2(u)(x \circ a) - [d_2(u), a]x + u(d_2(x) \circ a) - [u, a]d_2(x) = 0$$

for all $x, u \in R$. It means

$$(d_1 + d_2)(u)(x \circ a) - [u, a](d_1 + d_2)(x) - d_1([u, a])x - [d_2(u), a]x = 0 \tag{61}$$

for all $x, u \in R$. Replacing x by xr in the last equation, we get

$$(d_1 + d_2)(u)(x \circ a)r + (d_1 + d_2)(u)x[r, a] - [u, a](d_1 + d_2)(x)r \tag{62}$$

$$-[u, a]x(d_1 + d_2)(r) - d_1([u, a])xr - [d_2(u), a]xr = 0 \quad \text{for all} \ \ x, u, r \in R.$$

Right multiplying Eq. (61) by r, and using the last equation we get

$$(d_1 + d_2)(u)x[r, a] - [u, a]x(d_1 + d_2)(r) = 0 \quad \text{for all} \ \ x, u, r \in R. \tag{63}$$

In view of ([7], Lemma 1.3.2), there exists $\lambda \in \mathcal{C}$ such that

$$(d_1 + d_2)(x) = \lambda[x, a] \quad \text{for all} \ \ x \in R. \tag{64}$$

And Eq. (60) becomes

$$d_1(x \circ a) - d_1(x) \circ a + \lambda[x, a^2] = 0 \quad \text{for all} \ \ x \in R. \tag{65}$$

Note that, if we replace x by a in the last equation and by Leibniz's rule, we get $d_1(a^2) = d_1(a) \circ a = 0$.
Now, we replace x by xa in Eq. (65) we get

$$d_1(x(a \circ a)) - d_1([x, a]a) - d_1(x)(a \circ a) + [d_1(x), a]a + [x, a]d_1(a) + \lambda[x, a^2]a = 0 \tag{66}$$

for all $x \in R$. Which implies

$$\lambda[x, a^2]a - [x, d_1(a)]a = 0 \quad \text{for all} \ \ x \in R. \tag{67}$$

Since $h : R \to R$, $h(x) = \lambda[x, a^2] - [x, d_1(a)]$ define a derivation, by [12, Lemma 1], we have

$$\lambda[x, a^2] - [x, d_1(a)] = 0 \quad \text{for all} \ \ x \in R. \tag{68}$$

Combining last equation with Eq. (65), and after simplification we get

$$2xd_1(a) = 0 \quad \text{for all} \ \ x \in R. \tag{69}$$

Which implies $d_1(a) = 0$ and by eq. (68) we get $a^2 \in Z(R)$.
Now, we assume $Z(R) \neq (0)$ and

$$d_1(x \circ a) + d_2(x) \circ a \in Z(R) \quad \text{for all} \ \ x \in R. \tag{70}$$

By putting $d = d_1 + d_2$ and $b = d_1(a)$, the last equation becomes

$$d(x) \circ a + x \circ b \in Z(R) \quad \text{for all} \quad x \in R. \tag{71}$$

Let $z \neq 0$ be in $Z(R)$, then by replacing x by xz in Eq. (70) and using it, we get

$$(x \circ a)d(z) \in Z(R) \quad \text{for all} \quad x \in R. \tag{72}$$

So $x \circ a \in Z(R)$ for all $x \in R$ or $d(z) = 0$ for all $z(\neq 0) \in Z(R)$. If $x \circ a \in Z(R)$, for all $x \in R$ then $a \in Z(R)$ and so $x \in Z(R)$ for all $x \in R$, which means that R is commutative, a contradiction. Hence, we must have $d(z) = 0$ for all $z \in Z(R)$. Letting $x = z$ in Eq. (71) and since R is 2-torsion free we get

$$d(z)a + zb \in Z(R) \quad \text{for all} \quad x \in R. \tag{73}$$

Which implies $zb \in Z(R)$ and by Fact (1.4) we have $b \in Z(R)$. And Eq. (71) becomes

$$d(x) \circ a + 2xb \in Z(R) \quad \text{for all} \quad x \in R. \tag{74}$$

Commute this expression with xb to obtain

$$[d(x) \circ a, x]b = 0 \quad \text{for all} \quad x \in R. \tag{75}$$

Then either $[d(x) \circ a, x] = 0$ for all $x \in R$, or $b = 0$. Remarking that even in the last case, we have $[d(x) \circ a, x] = 0$ for all $x \in R$ and by using Fact (2), we obtain the desired result. ∎

The following example shows that the primeness hypothesis in Theorem 1 is not superfluous.

Example. We consider the ring $R = \mathcal{R}[X] \times M_2(\mathcal{R})$, which is a noncommutative semiprime ring. Consider the derivation d of R defined by $d(P, M) = (P', 0)$ for all $(P, M) \in R$ with P' is the usual derivation of the polynomial P.
Set $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. For $a = (0, A)$, and $d_1 = d_2 = d$ we have that $d_1(xa) - d_2(ax) = 0$ for all $x \in R$, but $a \notin Z(R)$.

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